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Stability of functional differential equations with impulses ☆

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Abstract

In this paper, the stability of functional differential equations (FDE) with impulses is investigated. Some comparison theorems are given. Several Lyapunov–Razumikhin functions of partial components of the state variable x, which can be much easier constructed, are used so that the conditions ensuring that stability are simpler and less restrictive. The results improve and generalize the ones in the literature. An example is also given to illustrate the importance of our results.

Keywords: FDE with impulses; Uniform stability; Uniform asymptotic stability; Lyapunov–Razumikhin functions

1. Introduction

It is well known that the method of Lyapunov–Razumikhin functions has been very powerful and effective in the study of stability for FDE as well as for FDE with impulses [1–3]. However, we always put all components of the state variable x together into one function $V(t,x)$ and then impose certain conditions on $V(t,x)$, $D^+V(t,x)$ and $V(t_k, x + I_k(x))$ (where t_k is impulsive point) to guarantee the required stability. Unfortunately, to construct such functions is rather difficult. This restricts the applications of Lyapunov–Razumikhin function method. To overcome this difficulty, [1] develop a new technique in studying stability of FDE with impulses, in which

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the components $x_1, x_2, \dots, x_n$ of x are divided into two groups; correspondingly, two functions $V_1(t, x_1, \dots, x_m)$ and $V_2(t, x_{m+1}, \dots, x_n)$ are adopted, then according to the cases of $V_1 \geqslant V_2$ or $V_1 \leqslant V_2$ and $V_1(t_k^-) \geqslant V_2(t_k^-)$ or $V_1(t_k^-) \leqslant V_2(t_k^-)$ certain conditions on D^+V_1 or D^+V_2 and $V_1(t_k)$ or $V_2(t_k)$ are imposed to guarantee the required stability. In this way, to construct the suitable functions is rather easy and the obtained conditions are less restrictive. Furthermore, the components of x can be actually divided into more groups and correspondingly, more functions can be adopted. Therefore, this technique is rather flexible.

In this paper, we extend this technique successfully. We establish some comparison theorems using several Lyapunov–Razumikhin functions of partial components of the state variable x. Employing these comparison theorems, we obtain several stability theorems. The theorems in [1] are included as special cases of our results.

In the end, an example is given to illustrate the advantages of the obtained results.

2. Preliminaries

We consider the FDE with impulses

$$\begin{cases} x' = f(t, x_t), & t \neq t_k, \\ x(t_k) = x\big(t_k^-\big) + I_k\big(x\big(t_k^-\big)\big), & k \in Z^+, \end{cases} \tag{1}$$

where Z^+ is the set of all positive integers, $f \in C(R^+ \times PC, R^n)$, $R^+ = [0, \infty)$, $PC = PC([-\tau, 0], R^n)$, where $\tau > 0$, and $PC([-\tau, 0], R^n) = \{\varphi : [-\tau, 0] \to R^n,\ \varphi(t)$ is continuous everywhere except at a finite number of points $\bar{t}$ at which $\varphi(\bar{t}^+)$ and $\varphi(\bar{t}^-)$ exist and $\varphi(\bar{t}^+) = \varphi(\bar{t})\}$. $I_k \in C(R^n, R^n)$ for $k \in Z^+$. $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ with $t_k \to \infty$ as $k \to \infty$ and $x'(t)$ denotes the right-hand derivative of $x(t)$. $x_t \in PC$ is defined by $x_t(s) = x(t+s)$, $-\tau \leqslant s \leqslant 0$. For $\varphi \in PC$, the norm of φ is defined by $\|\varphi\| = \sup_{-\tau \leqslant s \leqslant 0} |\varphi(s)|$, where $|\cdot|$ denotes the norm of vector in R^n.

Throughout this paper, we always assume that $f(t, \varphi)$ satisfies certain conditions to ensure the global existence and uniqueness of solutions of (1) (cf. [4]). The unique solution of (1) through $(t_0, \varphi) \in R^+ \times PC$ defined on $[t_0 - \tau, \infty)$ is denoted by $x(t, t_0, \varphi)$. Furthermore, we assume $f(t, 0) \equiv 0$ and $I_k(0) = 0$ so that (1) has solution $x(t) \equiv 0$, which is called the zero solution.

The function $V : R^+ \times R^m \to R^+$ belongs to $V^m(\cdot)$ if

(1) the function V is continuous on $[t_{k-1}, t_k) \times R^m$, $k \in Z^+$, and $V(t, 0) \equiv 0$;
(2) $V(t, x)$ is locally Lipschitzian in $x \in R^m$;
(3) for each $k \in Z^+$, the following limits exist finitely:

$$\lim_{(t,y) \to (t_k^-, x)} V(t, y) = V\big(t_k^-, x\big).$$

Let $V \in V^m(\cdot)$, for $(t, x) \in [t_{k-1}, t_k) \times R^m$, $k \in Z^+$, D^+V along the solution $x(t)$ of (1) is defined as

$$D^+V\big(t, x(t)\big) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta}\big[V\big(t+\delta, x(t+\delta)\big) - V\big(t, x(t)\big)\big].$$

For any $\rho > 0$, let $PC(\rho) = \{\varphi \in PC\colon\ \|\varphi\| < \rho\}$.

Let

$$K_0 = \big\{a(u) \in C[R^+, R^+],\ \text{increasing},\ a(0) = 0\big\},$$

$$K_1 = \big\{a(u) \in K_0,\ \text{strictly increasing},\ a(u) \geqslant u\big\}.$$

Definition. (1) is said to be

(1) uniformly stable, if for any $t_0 \in R^+$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\varphi \in PC(\delta)$ implies $|x(t, t_0, \varphi)| < \varepsilon$, $t \geqslant t_0$.
(2) uniformly asymptotically stable, if it is uniformly stable and there exists $\delta > 0$ such that for any $\varepsilon > 0$ there is $T = T(\varepsilon) > 0$ such that $\varphi \in PC(\delta)$ implies $|x(t, t_0, \varphi)| < \varepsilon$ for $t \geqslant t_0 + T$.

In the following we will split $x = (x_1, x_2, \ldots, x_n)^T$ into several vectors, say, $(x_1^{(1)}, \ldots, x_{n_1}^{(1)})^T$, $(x_1^{(2)}, \ldots, x_{n_2}^{(2)})^T, \ldots, (x_1^{(m)}, \ldots, x_{n_m}^{(m)})^T$ such that $n_1 + n_2 + \cdots + n_m = n$ and

$$x = \left(x_1^{(1)}, \ldots, x_{n_1}^{(1)}, x_1^{(2)}, \ldots, x_{n_2}^{(2)}, \ldots, x_1^{(m)}, \ldots, x_{n_m}^{(m)}\right)^T.$$

For the sake of convenience, we denote

$$x^{(j)} = \left(x_1^{(j)}, \ldots, x_{n_j}^{(j)}\right), \quad j = 1, 2, \ldots, m, \quad \text{and} \quad x = \left(x^{(1)}, x^{(2)}, \ldots, x^{(m)}\right)^T.$$

Note the order of components in $x^{(j)}$ is not necessarily the same as in x.

Let

$$\left|x^{(j)}\right| = \max_{1 \leqslant k \leqslant n_j} \left|x_k^{(j)}\right|, \quad j = 1, 2, \ldots, m,$$

and thus

$$|x| = \max_{1 \leqslant j \leqslant m} \left|x^{(j)}\right|.$$

3. Main results

For the sake of simplicity, we start with the case of $m = 2$ and first establish the following comparison theorem on uniform stability.

Theorem 1. *Suppose there exist $V_1(t, x^{(1)}) \in V^{n_1}(\cdot)$ and $V_2(t, x^{(2)}) \in V^{n_2}(\cdot)$, $n = n_1 + n_2$, such that*

(i) $a_j(|x^{(j)}|) \leqslant V_j(t, x^{(j)}) \leqslant b_j(|x^{(j)}|)$, $a_j, b_j \in K_0$, $j = 1, 2$;
(ii) *when $V_1(t) \geqslant V_2(t)$, there holds*

$$D^+ V_1(t) \leqslant g\big(t, V_1(t)\big) \quad \text{if } V_1(t) \geqslant V_1(s),\ t - \tau \leqslant s \leqslant t;$$

when $V_1(t) \leqslant V_2(t)$, there holds

$$D^+ V_2(t) \leqslant g\big(t, V_2(t)\big) \quad \text{if } V_2(t) \geqslant V_2(s),\ t - \tau \leqslant s \leqslant t,$$

where $V_j(t) = V_j(t, x^{(j)}(t))$, $j = 1, 2$, $x(t) = (x^{(1)}(t), x^{(2)}(t))$ is the solution of (1) *and $g : R^+ \times R^+ \to R^+$ is continuous, $g(t, \cdot) \in K_0$ for each t;*
(iii) *for all $k \in Z^+$, when $V_1(t_k^-) \geqslant V_2(t_k^-)$, there holds*

$$\max\big\{V_1(t_k), V_2(t_k)\big\} \leqslant G_k\big(V_1\big(t_k^-\big)\big);$$

when $V_1(t_k^-) \leqslant V_2(t_k^-)$, there holds

$$\max\big\{V_1(t_k), V_2(t_k)\big\} \leqslant G_k\big(V_2\big(t_k^-\big)\big),$$

where $G_k \in K_1$;

(iv) *the zero solution* $y(t) \equiv 0$ *of*

$$\begin{cases} y' = g(t, y), & t \neq t_k, \\ y(t_k) = G_k\big(y\big(t_k^-\big)\big), & k \in Z^+, \end{cases} \tag{2}$$

is uniformly stable.

Then (1) *is uniformly stable.*

Proof. Let $t_0 \in R_+$ be given. Without loss of generality, we assume that $t_0 < t_k$, $k \in Z^+$. Suppose $x(t) = x(t, t_0, \varphi)$ is the solution of (1). Define a function $V(t)$ as follows:

$$V(t) = \begin{cases} V_1(t), & V_1(t) \geqslant V_2(t), \\ V_2(t), & V_2(t) \geqslant V_1(t). \end{cases} \tag{3}$$

We point out first that for any $t \geqslant t_0 - \tau$,

$$\big[a_1\big(\big|x^{(1)}(t)\big|\big) + a_2\big(\big|x^{(2)}(t)\big|\big)\big]/2 \leqslant V(t) \leqslant b_1\big(\big|x^{(1)}(t)\big|\big) + b_2\big(\big|x^{(2)}(t)\big|\big). \tag{4}$$

In fact, if $V_1(t) \geqslant V_2(t)$, then by (3) and assumption (i),

$$V(t) = V_1(t) \geqslant \big[V_1(t) + V_2(t)\big]/2 \geqslant \big[a_1\big(\big|x^{(1)}(t)\big|\big) + a_2\big(\big|x^{(2)}(t)\big|\big)\big]/2.$$

Whereas if $V_1(t) \leqslant V_2(t)$, then

$$V(t) = V_2(t) \geqslant \big[V_1(t) + V_2(t)\big]/2 \geqslant \big[a_1\big(\big|x^{(1)}(t)\big|\big) + a_2\big(\big|x^{(2)}(t)\big|\big)\big]/2.$$

It is obvious that

$$V(t) \leqslant V_1(t) + V_2(t) \leqslant b_1\big(\big|x^{(1)}(t)\big|\big) + b_2\big(\big|x^{(2)}(t)\big|\big).$$

Next, we claim that, for $t \geqslant t_0$,

$$\begin{aligned} &D^+V(t) \leqslant g\big(t, V(t)\big) \quad \text{if } V(t) \geqslant V(s),\ t - \tau \leqslant s \leqslant t,\ t \neq t_k, \\ &V(t_k) \leqslant G_k\big(V\big(t_k^-\big)\big), \quad k \in Z^+. \end{aligned} \tag{5}$$

In fact, suppose $V_1(t_0) \geqslant V_2(t_0)$ and there exists $r_1 > t_0$, $V_1(t) \geqslant V_2(t)$, $t \in [t_0, r_1)$. By (3), we get $V(t) = V_1(t)$ for $t \in [t_0, r_1)$.

Case 1. If $t = t_k$ for some $k \in Z^+$, then by (iii) $V(t_k) = V_1(t_k) \leqslant G_k(V(t_k^-))$.

Case 2. t is not a time of impulse effect and $V(t) \geqslant V(s)$, $t - \tau \leqslant s \leqslant t$. Then if $V_1(s) \leqslant V_2(s)$ we have $V(s) = V_2(s)$. Clearly, $V(s) \leqslant V(t)$ implies $V_1(s) \leqslant V_2(s) = V(s) \leqslant V(t) = V_1(t)$.

If $V_1(s) \geqslant V_2(s)$ we have $V(s) = V_1(s)$. Obviously, $V(s) \leqslant V(t)$ implies $V_1(s) = V(s) \leqslant V(t) = V_1(t)$. In conclude, $V(s) \leqslant V(t)$, $t - \tau \leqslant s \leqslant t$, $t \neq t_k$, implies $V_1(s) \leqslant V_1(t)$, $t - \tau \leqslant s \leqslant t$, $t \neq t_k$. So by (ii) we have $D^+V(t) = D^+V_1(t) \leqslant g(t, V(t))$.

If $r_1 = \infty$ we arrive at the assertion that (5) is true for all $t \geqslant t_0$. Otherwise there exists $r_2 \geqslant r_1$ such that $V_1(t) \leqslant V_2(t)$, $t \in [r_1, r_2)$. When $r_1 = t_i$ for some $i \in Z^+$ we have $V_1(t_i^-) \geqslant V_2(t_i^-)$ and $V_1(t_i) \leqslant V_2(t_i)$. In this case, by (iii) we have $V(t_i) \leqslant G_i(V_1(t_i^-)) = G_i(V(t_i^-))$. When $r_1 \neq t_i$, we set $V(t) = V_2(t)$ for $t \in [r_1, r_2)$.

Case 1. If $t = t_k$ for some $k \in Z^+$, then by (iii) $V(t_k) = V_2(t_k) \leqslant G_k(V(t_k^-))$.

Case 2. t is not a time of impulse effect and $V(t) \geqslant V(s)$, $t - \tau \leqslant s \leqslant t$. Then if $V_2(s) \leqslant V_1(s)$ we have $V(s) = V_1(s)$. Clearly, $V(s) \leqslant V(t)$ implies $V_2(s) \leqslant V_1(s) = V(s) \leqslant V(t) = V_2(t)$.

If $V_2(s) \geqslant V_1(s)$ we have $V(s) = V_2(s)$. Obviously, $V(s) \leqslant V(t)$ implies $V_2(s) = V(s) \leqslant V(t) = V_2(t)$. In conclude, $V(s) \leqslant V(t)$, $t - \tau \leqslant s \leqslant t$, $t \neq t_k$, implies $V_2(s) \leqslant V_2(t)$, $t - \tau \leqslant s \leqslant t$, $t \neq t_k$. So by (ii) we have $D^+V(t) = D^+V_2(t) \leqslant g(t, V(t))$.

If $r_2 = \infty$ then (5) holds for all $t \geqslant t_0$. Otherwise repeat the above argument to arrive at the assertion that (5) is valid for all $t \geqslant t_0$. As for the case of $V_1(t) \leqslant V_2(t)$ for $t \in [t_0, r_1)$, the process is similar and we omit it.

We are now in a position to show the uniform stability of (1).

Let $y(t) = y(t, t_0, y_0)$ $(y_0 > 0)$ be the maximal solution of (2). Choose $\delta > 0$ such that

$$b_1(\delta) + b_2(\delta) < y_0.$$

We shall prove that, if $\|\varphi\| < \delta$,

$$V(t) < y(t), \quad t \geqslant t_0. \tag{6}$$

By (4),

$$V(t) \leqslant b_1(\delta) + b_2(\delta) < y_0, \quad t \in [t_0 - \tau, t_0].$$

Let $h_0 = b_1(\delta) + b_2(\delta)$, to prove that (6) holds for $t \in [t_0, t_1)$, we prove that

$$y(t) - V(t) \geqslant y_0 - h_0 \quad \text{for } t \in [t_0, t_1). \tag{7}$$

If this is not true, then there exists a solution $x(t) = x(t, t_0, \varphi)$ with $\|\varphi\| < \delta$ and $t_0 \leqslant \underline{t} < \bar{t} < t_1$ satisfying

(a) $y(\bar{t}) - V(\bar{t}) < y_0 - h_0$;
(b) $V(\underline{t}) = h_0$, $V(t) \geqslant V(t+s)$ for $s \in [-\tau, 0]$, $t \in [\underline{t}, \bar{t}]$; and
(c) $V(t) \leqslant y(t)$, $t \in [\underline{t}, \bar{t}]$.

By (b), (c) and (5), we have

$$D^+V(t) \leqslant g\big(t, V(t)\big) \leqslant g\big(t, y(t)\big) = y'(t), \quad t \in [\underline{t}, \bar{t}].$$

Integrating the above inequality from $\underline{t}$ to $\bar{t}$, we get

$$V(\bar{t}) \leqslant V(\underline{t}) + y(\bar{t}) - y(\underline{t}) \leqslant h_0 + y(\bar{t}) - y_0,$$

which contradicts (a). Let $h_1 = y_0 - h_0$, by (7),

$$V(t) \leqslant y(t) - h_1 \leqslant y(t_1) - h_1, \quad t \in [t_0, t_1).$$

Then $V(t_1^-) \leqslant y(t_1^-) - h_1 < y(t_1^-)$ and

$$V(t_1) \leqslant G_1\big(V\big(t_1^-\big)\big) < G_1\big(y\big(t_1^-\big)\big) = y(t_1).$$

Next, we prove that

$$y(t) - V(t) \geqslant y(t_1) - \max\big\{V(t_1), y(t_1) - h_1\big\}.$$

Otherwise, there exist $t_1 \leqslant \underline{t}' < \bar{t}' < t_2$ satisfying

(a)′ $y(\bar{t}') - V(\bar{t}') < y(t_1) - \max\{V(t_1), y(t_1) - h_1\}$;
(b)′ $V(\underline{t}') = \max\{V(t_1), y(t_1) - h_1\}$, $V(t) \geqslant V(t+s)$ for $s \in [-\tau, 0]$, $t \in [\underline{t}', \bar{t}']$; and
(c)′ $V(t) \leqslant y(t)$, $t \in [\underline{t}', \bar{t}']$.

Using the same proof as for $t \in [t_0, t_1)$, we can get a contradiction. By induction, (6) is correct.

From the uniform stability of the zero solution of (2), given $\varepsilon > 0$, there exists $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$ such that

$$y(t) < \min\big\{a_1(\varepsilon)/2, a_2(\varepsilon)/2\big\} \quad \text{provided } y_0 < \bar{\delta}.$$

Therefore, if $\|\varphi\| < \delta$,

$$V(t) < \min\{a_1(\varepsilon)/2, a_2(\varepsilon)/2\} \quad \text{provided } y_0 < \bar{\delta}.$$

Together with (4), we get

$$|x^{(i)}(t)| < \varepsilon, \quad t \geqslant t_0,\ i = 1, 2.$$

Thus we have $|x(t)| < \varepsilon$, $t \geqslant t_0$. The proof is complete. □

From Theorem 1, we have the following corollary which is Theorem 1 in [1].

Corollary 1. *Suppose $g \equiv 0$, $G_k(s) = (1 + d_k)s$ for $s \geqslant 0$, $d_k \geqslant 0$ and $\sum_{k=1}^{\infty} d_k < \infty$. Then* (1) *is uniformly stable.*

The next result is on the uniformly asymptotic stability of (1).

Theorem 2. *Suppose there exist $V_1(t, x^{(1)}) \in V^{n_1}(\cdot)$ and $V_2(t, x^{(2)}) \in V^{n_2}(\cdot)$, $n = n_1 + n_2$, such that*

(i) $a_j(|x^{(j)}|) \leqslant V_j(t, x^{(j)}) \leqslant b_j(|x^{(j)}|)$, $a_j, b_j \in K_0$, $j = 1, 2$;

(ii) *for all $k \in Z^+$, when $V_1(t_k^-) \geqslant V_2(t_k^-)$, there holds*

$$\max\{V_1(t_k), V_2(t_k)\} \leqslant (1 + d_k)V_1(t_k^-);$$

when $V_1(t_k^-) \leqslant V_2(t_k^-)$, there holds

$$\max\{V_1(t_k), V_2(t_k)\} \leqslant (1 + d_k)V_2(t_k^-),$$

where $d_k \geqslant 0$, $\sum_{k=1}^{\infty} d_k < \infty$;

(iii) *when $V_1(t) \geqslant V_2(t)$, there holds*

$$D^+V_1(t) \leqslant -F_1(t, |x^{(1)}(t)|) + g(t)H(V_1(t)), \quad P_1(V_1(t)) > V_1(s),\ t - \tau \leqslant s \leqslant t;$$

when $V_1(t) \leqslant V_2(t)$, there holds

$$D^+V_2(t) \leqslant -F_2(t, |x^{(2)}(t)|) + g(t)H(V_2(t)), \quad P_2(V_2(t)) > V_2(s),\ t - \tau \leqslant s \leqslant t,$$

where $x(t) = (x^{(1)}(t), x^{(2)}(t))$ is the solution of (1), *$P_i, g : R^+ \to R^+$ are continuous, for $s > 0$, $P_i(s) > Ms$ $(i = 1, 2)$, $M = \prod_{k=1}^{\infty}(1 + d_k)$, $\int_0^{\infty} g(t)\,\mathrm{d}t < \infty$, $H \in K_0$, $F_i : R^+ \times R^+ \to R^+$ $(i = 1, 2)$ are continuous, for $\alpha \geqslant s \geqslant \sigma > 0$, $F_i(t, s) \geqslant \psi(t, \alpha, \sigma) > 0$, where $\psi(t, \alpha, \sigma)$ is continuous and given $\beta > 0$, there exists $\tilde{T} = \tilde{T}(\beta, \alpha, \sigma) > 0$ such that $\int_T^{T+\tilde{T}} \psi(t, \alpha, \sigma)\,\mathrm{d}t > (M' + 1)\beta$ for any $T > 0$, where $M' = \sum_{k=1}^{\infty} d_k$;*

(iv) *the zero solution $y(t) \equiv 0$ of*

$$\begin{cases} y' = g(t)H(y), & t \neq t_k, \\ y(t_k) = (1 + d_k)y(t_k^-), & k \in Z^+, \end{cases}$$

is uniformly stable.

Then (1) *is uniformly asymptotically stable.*

Proof. Let $t_0 \in R_+$ be given. Without loss of generality, we assume that $t_0 < t_k$, $k \in Z^+$. As in the proof of Theorem 1, suppose $x(t) = x(t, t_0, \varphi)$ is the solution of (1). If we define $V(t)$ by (3), then (4) is still valid. Let $P(s) = \min\{P_1(s), P_2(s)\}$. Clearly, $P(s)$ is also continuous and $P(s) > Ms$ for $s > 0$.

Next we claim that

(a) on any interval where $V_1(t) \geqslant V_2(t)$, $t \neq t_k$, $k \in Z^+$, there holds

$$D^+V(t) \leqslant -F_1\big(t, \big|x^{(1)}(t)\big|\big) + g(t)H\big(V(t)\big) \quad \text{if } V(s) < P\big(V(t)\big) \text{ for } s \in [t-\tau, t]; \tag{8}$$

(b) on any interval where $V_1(t) \leqslant V_2(t)$, $t \neq t_k$, $k \in Z^+$, there holds

$$D^+V(t) \leqslant -F_2\big(t, \big|x^{(2)}(t)\big|\big) + g(t)H\big(V(t)\big) \quad \text{if } V(s) < P\big(V(t)\big) \text{ for } s \in [t-\tau, t]; \tag{9}$$

(c) $V(t_k) \leqslant (1 + d_k)V(t_k^-)$, $k \in Z^+$.

In fact, suppose $V_1(t_0) \geqslant V_2(t_0)$ and there exists $r_1 > t_0$, $V_1(t) \geqslant V_2(t)$, $t \in [t_0, r_1)$. By (3), we get $V(t) = V_1(t)$ for $t \in [t_0, r_1)$.

Case 1. If $t = t_k$ for some $k \in Z^+$, then by (ii) $V(t_k) = V_1(t_k) \leqslant (1 + d_k)V(t_k^-)$.

Case 2. $t \neq t_k$ and $P(V(t)) > V(s)$, $t - \tau \leqslant s \leqslant t$. Then if $V_1(s) \leqslant V_2(s)$ we have $V(s) = V_2(s)$. Clearly, $V(s) < P(V(t))$ implies $V_1(s) \leqslant V_2(s) = V(s) < P(V(t)) = P(V_1(t))$.

If $V_1(s) \geqslant V_2(s)$ we have $V(s) = V_1(s)$. Obviously, $V(s) < P(V(t))$ implies $V_1(s) = V(s) < P(V(t)) = P(V_1(t))$. In conclude, $V(s) < P(V(t))$, $t - \tau \leqslant s \leqslant t$, $t \neq t_k$, implies $V_1(s) < P(V_1(t))$, $t - \tau \leqslant s \leqslant t$, $t \neq t_k$. So by (iii) we have $D^+V(t) = D^+V_1(t) \leqslant -F_1(t, x^{(1)}(t)) + g(t)H(V(t))$ if $V(s) < P(V(t))$, $s \in [t - \tau, t]$.

If $r_1 = \infty$ we arrive at the assertion that (8) is true for all $t \geqslant t_0$. Otherwise there exists $r_2 \geqslant r_1$ such that $V_1(t) \leqslant V_2(t)$ for $t \in [r_1, r_2)$. When $r_1 = t_i$ for some $i \in Z^+$ we have $V_1(t_i^-) \geqslant V_2(t_i^-)$ and $V_1(t_i) \leqslant V_2(t_i)$. In this case, by (ii) we have $V(t_i) \leqslant (1 + d_i)V_1(t_i^-) = (1 + d_i)V(t_i^-)$. When $r_1 \neq t_i$, we set $V(t) = V_2(t)$ for $t \in [r_1, r_2)$. In a similar way, we can show that for $t \in [r_1, r_2)$,

$$D^+V(t) \leqslant -F_2\big(t, \big|x^{(2)}(t)\big|\big) + g(t)H\big(V(t)\big) \quad \text{if } V(s) < P\big(V(t)\big) \text{ for } s \in [t-\tau, t].$$

If $r_2 = \infty$ then (9) holds for all $t \geqslant r_1$. Otherwise we may continue the above process and we can see that the interval $[t_0, \infty)$ can be divided into finite or infinite number of successive subintervals and on each of them either (8) or (9) holds and (c) is always true. As for the case of $V_1(t) \leqslant V_2(t)$ for $t \in [t_0, r_1)$, the process is similar.

Since $V_i(t) \geqslant V_i(s)$, $t - \tau \leqslant s \leqslant t$ $(i = 1, 2)$, imply $P_i(V_i(t)) > V_i(s)$, $t - \tau \leqslant s \leqslant t$, it is obvious that (1) is uniformly stable by Theorem 1.

Furthermore, we show the uniformly asymptotic stability.

For $\varepsilon = \varepsilon_0$, we find the corresponding $\delta_0 > 0$ such that $|x(t, t_0, \varphi)| < \varepsilon_0$, $V(t) \leqslant A \triangleq \frac{1}{2}\min\{a_1(\varepsilon_0), a_2(\varepsilon_0)\}$, $\|\varphi\| < \delta_0$, $t \geqslant t_0$. Given $\varepsilon > 0$ with $\varepsilon < \varepsilon_0$, let $B = \max_{0 \leqslant s \leqslant A} H(s)$, $\varepsilon^* \triangleq \frac{1}{2}\min\{a_1(\varepsilon), a_2(\varepsilon)\}$, $0 < d < \min_{M^{-1}\varepsilon^* \leqslant s \leqslant A}\{P(s) - Ms\}$ and $d < \varepsilon^*$. Let $N = N(\varepsilon)$ be the smallest positive integer such that $A \leqslant \varepsilon^* + Nd$. Since $\int_0^\infty g(t)\,\mathrm{d}t < \infty$, there must exists $T > 0$ such that for $t \geqslant T$, we have $B\int_T^t g(s)\,\mathrm{d}s < M^{-1}d/6$. Next we prove that there exists $T_1 \geqslant T$ such that

$$V(T_1) < M^{-1}\big[\varepsilon^* + (N-1)d\big].$$

Otherwise, for $t \geqslant T$,

$$V(t) \geqslant M^{-1}\big[\varepsilon^* + (N-1)d\big].$$

Therefore,

$$P\big(V(t)\big) > MV(t) + d \geqslant \varepsilon^* + Nd \geqslant A \geqslant V(t+s), \quad -\tau \leqslant s \leqslant 0.$$

Let $I_1 = \{t\colon\ V_1(t) \geqslant V_2(t)\}$, $I_2 = \{t\colon\ V_1(t) \leqslant V_2(t)\}$. Then $t \in I_1$ implies

$$b_1\big(\big|x^{(1)}(t)\big|\big) \geqslant V_1(t) = V(t) \geqslant \varepsilon^*/M.$$

Because b_1 is increasing, there exists $\sigma_1 > 0$ such that $|x^{(1)}(t)| \geqslant \sigma_1$, $t \in I_1$. Similarly we can prove that there exists $\sigma_2 > 0$ such that $|x^{(2)}(t)| \geqslant \sigma_2$, $t \in I_2$.

Then, by (8) and (9), we obtain

$$D^+V(t) \leqslant -\psi(t, \varepsilon_0, \sigma) + g(t)H\big(V(t)\big) \quad \text{for } t \in t \geqslant T,\ \sigma = \min\{\sigma_1, \sigma_2\}.$$

From condition (iii), there exists $\tilde{T} = \tilde{T}(A, \varepsilon_0, \sigma) > 0$ such that

$$\begin{aligned} V(T+\tilde{T}) &\leqslant V(T) - \int_T^{T+\tilde{T}} \psi(t, \varepsilon_0, \sigma)\, \mathrm{d}t + B\int_T^{T+\tilde{T}} g(t)\, \mathrm{d}t + \sum_{T<t_j\leqslant T+\tilde{T}} \big[V(t_j) - V\big(t_j^-\big)\big] \\ &\leqslant A(1+M') - \int_T^{T+\tilde{T}} \psi(t, \varepsilon_0, \sigma)\, \mathrm{d}t < 0. \end{aligned}$$

This contradicts $V(t) \geqslant 0$. So we can choose $T_1 = T + \tilde{T}$.

Next, we claim that

$$V(t) < \big[\varepsilon^* + (N-1)d\big] + d/2 \quad \text{for } t \geqslant T_1.$$

Suppose $T_1 \in [t_{j-1}, t_j)$, we first prove that

$$V(t) < M^{-1}\big[\varepsilon^* + (N-1)d\big] + M^{-1}d/6 \quad \text{for } t \in [T_1, t_j). \tag{10}$$

If (10) is not true, there must exist $T_1 < \bar{t}_1 < \bar{t}_2 < t_j$ such that

$$V(\bar{t}_1) = M^{-1}\big[\varepsilon^* + (N-1)d\big], \tag{11}$$

$$V(\bar{t}_2) = M^{-1}\big[\varepsilon^* + (N-1)d\big] + M^{-1}d/6 \tag{12}$$

and

$$V(\bar{t}_1) \leqslant V(t) \leqslant V(\bar{t}_2), \quad t \in [\bar{t}_1, \bar{t}_2]. \tag{13}$$

From (11) and (13),

$$\begin{aligned} P\big(V(t)\big) &> MV(t) + d \geqslant MV(\bar{t}_1) + d = \varepsilon^* + Nd \\ &\geqslant A \geqslant V(t+s), \quad -\tau \leqslant s \leqslant 0,\ \bar{t}_1 \leqslant t \leqslant \bar{t}_2. \end{aligned}$$

Together with (8) and (9), it follows that

$$V(\bar{t}_2) \leqslant V(\bar{t}_1) + B\int_{\bar{t}_1}^{\bar{t}_2} g(t)\, \mathrm{d}t < M^{-1}\big[\varepsilon^* + (N-1)d\big] + M^{-1}d/6,$$

which contradicts (12). Then we get

$$V(t_j) \leqslant (1+d_j)V\left(t_j^-\right) \leqslant (1+d_j)\left\{M^{-1}\left[\varepsilon^* + (N-1)d\right] + M^{-1}d/6\right\}.$$

Denote $\mu_m = \int_{t_m}^{t_{m+1}} g(t)\,\mathrm{d}t$, $m \geqslant j$. Then $\mu_m \geqslant 0$, $B\sum_{m=j}^{\infty}\mu_m < M^{-1}d/6$. Let $\{\nu_m\}$, $m \geqslant j$, be a sequence, satisfying $\nu_m > 0$, $\sum_{m=j}^{\infty}\nu_m < M^{-1}d/6$. In a similar way as in the proof of (10), we can prove that

$$V(t) < (1+d_j)\left\{M^{-1}\left[\varepsilon^* + (N-1)d\right] + M^{-1}d/6\right\} + B\mu_j + \nu_j, \quad t \in [t_j, t_{j+1}).$$

By induction, we arrive at

$$V(t) < \prod_{k=j}^{l}(1+d_k)\left\{M^{-1}\left[\varepsilon^* + (N-1)d\right] + M^{-1}d/6\right\} + \prod_{k=j+1}^{l}(1+d_k)(B\mu_j + \nu_j)$$
$$+ \prod_{k=j+2}^{l}(1+d_k)(B\mu_{j+1} + \nu_{j+1}) + \cdots + (B\mu_l + \nu_l), \quad t \in [t_l, t_{l+1}),\ l \geqslant j.$$

Hence, by the definition of M,

$$V(t) < \varepsilon^* + (N-1)d + d/6 + M\sum_{k=j}^{\infty}(B\mu_k + \nu_k)$$
$$< \varepsilon^* + (N-1)d + d/2, \quad t \in [T_1, \infty).$$

Similarly, we can prove there exists $T_2 = T_1 + \hat{T}$, $\hat{T} = \max\{\tilde{T}, \tau\}$, such that

$$V(T_2) < M^{-1}\left[\varepsilon^* + (N-2)d + d/2\right],$$
$$V(t) < \varepsilon^* + (N-1)d, \quad t \geqslant T_2.$$

By induction, we obtain

$$V(t) < \varepsilon^*, \quad t \geqslant T + 2N\hat{T},$$

which, together with (4) and the definition of ε^*, yields

$$\left|x^{(1)}(t)\right| < \varepsilon, \quad \left|x^{(2)}(t)\right| < \varepsilon, \quad t \geqslant T + 2N\hat{T}.$$

Thus $|x(t)| < \varepsilon$, $t \geqslant t_0 + T + 2N\hat{T}$. The proof is complete. □

Corollary 2. *Suppose $F_i(t,s) \equiv \psi(t)w_i(s)$, where $\psi : R^+ \to R^+$ is continuous, given $\beta > 0$, there exists $\tilde{T} = \tilde{T}(\beta) > 0$ such that $\int_T^{T+\tilde{T}} \psi(t)\,\mathrm{d}t \geqslant \beta$ for any $T > 0$; $w_i : R^+ \to R^+$ are continuous and $w_i(s) > 0$ for $s > 0$, $i = 1, 2$. The other conditions of Theorem* 2 *hold. Then* (1) *is uniformly asymptotically stable.*

Remark. If $F_i(t,s) \equiv w_i(s)$, $g(t) \equiv 0$, Corollary 2 is just Theorem 2 in [1].

Furthermore, we may generalize the ideas behind Theorem 1 and 2 to obtain the following results.

Theorem 3. *Suppose there exist $V_j(t,x^{(j)}) \in V^{n_j}$, $j=1,2,\ldots,m$, with $\sum_{j=1}^m n_j = n$ such that*

(i) $a_j(|x^{(j)}|) \leqslant V_j(t,x^{(j)}) \leqslant b_j(|x^{(j)}|)$, $a_j, b_j \in K_0$, $j=1,2,\ldots,m$;

(ii) *when $V_j(t) = \max\{V_i(t)\colon 1 \leqslant i \leqslant m\}$, we have*

$$D^+V_j(t) \leqslant g\big(t, V_j(t)\big) \quad \text{if } V_j(t) \geqslant V_j(s),\ t-\tau \leqslant s \leqslant t,$$

where $V_j(t) = V_j(t, x^{(j)}(t))$, $j=1,2,\ldots,m$, $x(t) = (x^{(1)}(t), x^{(2)}(t), \ldots, x^{(m)}(t))$ is the solution of (1) *and $g: R^+ \times R^+ \to R^+$ is continuous, $g(t,0) \equiv 0$;*

(iii) *for all $k \in Z^+$, when $V_j(t_k^-) = \max_{1\leqslant i\leqslant m}\{V_i(t_k^-)\}$, there holds*

$$\max_{1\leqslant i\leqslant m}\big\{V_i(t_k)\big\} \leqslant G_k\big(V_j\big(t_k^-\big)\big),$$

where $G_k \in K_1$;

(iv) *the zero solution $y(t) \equiv 0$ of*

$$\begin{cases} y' = g(t,y), & t \neq t_k, \\ y(t_k) = G_k\big(y\big(t_k^-\big)\big), & k \in Z^+, \end{cases}$$

is uniformly stable.

Then (1) *is uniformly stable.*

Theorem 4. *Suppose there exist $V_j(t,x^{(j)}) \in V^{n_j}$, $j=1,2,\ldots,m$, with $\sum_{j=1}^m n_j = n$ such that*

(i) $a_j(|x^{(j)}|) \leqslant V_j(t,x^{(j)}) \leqslant b_j(|x^{(j)}|)$, $a_j, b_j \in K_0$, $j=1,2,\ldots,m$;

(ii) *for all $k \in Z^+$, when $V_j(t_k^-) = \max_{1\leqslant i\leqslant m}\{V_i(t_k^-)\}$, there holds*

$$\max_{1\leqslant i\leqslant m}\big\{V_i(t_k)\big\} \leqslant (1+d_k)V_j\big(t_k^-\big),$$

where $d_k \geqslant 0$, $\sum_{k=1}^\infty d_k < \infty$;

(iii) *when $V_j(t) = \max\{V_i(t)\colon 1 \leqslant i \leqslant m\}$, there holds*

$$D^+V_j(t) \leqslant -F_j\big(t, \big|x^{(j)}(t)\big|\big) + g(t)H\big(V_j(t)\big), \quad P_j\big(V_j(t)\big) > V_j(s),\ t-\tau \leqslant s \leqslant t,$$

where $x(t) = (x^{(1)}(t), x^{(2)}(t), \ldots, x^{(2)}(t))$ is the solution of (1), *$P_j, g: R^+ \to R^+$ are continuous, for $s > 0$, $P_j(s) > Ms$ $(j=1,2,\ldots,m)$, $M = \prod_{k=1}^\infty(1+d_k)$, $\int_0^\infty g(t)\,\mathrm{d}t < \infty$, $H \in K_0$, $F_j: R^+ \times R^+ \to R^+$, $j=1,2,\ldots,m$, are continuous, for $\alpha \geqslant s \geqslant \sigma > 0$, $F_j(t,s) \geqslant \psi(t,\alpha,\sigma) > 0$, where $\psi(t,\alpha,\sigma)$ is continuous and given $\beta > 0$, there exists $\tilde{T} = \tilde{T}(\beta,\alpha,\sigma) > 0$ such that $\int_T^{T+\tilde{T}} \psi(t,\alpha,\sigma)\,\mathrm{d}t \geqslant (M'+1)\beta$ for any $T > 0$, where $M' = \sum_{k=1}^\infty d_k$;*

(iv) *the zero solution $y(t) \equiv 0$ of*

$$\begin{cases} y' = g(t)H(y), & t \neq t_k, \\ y(t_k) = (1+d_k)y\big(t_k^-\big), & k \in Z^+, \end{cases}$$

is uniformly stable.

Then (1) *is uniformly asymptotically stable.*

4. Example

Consider the following equations:

$$\begin{cases} x_1' = -a_1(t)x_1(t) + a_2(t)x_2(t) + b_1(t)x_1\big(t - r_1(t)\big), & t \neq t_k, \\ x_2' = c_1(t)x_1(t) - c_2(t)x_2(t) + b_2(t)x_2\big(t - r_2(t)\big), & t \neq t_k, \\ x_1(t_k) = d_{k_1}x_1\big(t_k^-\big) + d_{k_2}x_2\big(t_k^-\big), & k \in Z^+, \\ x_2(t_k) = e_{k_1}x_1\big(t_k^-\big) + e_{k_2}x_2\big(t_k^-\big), & k \in Z^+, \end{cases} \tag{14}$$

where a_i, b_i, c_i, r_i $(i = 1, 2)$ are continuous functions on R^+, $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ and $\lim_{k\to\infty} t_k = \infty$, d_{k_i}, e_{k_i} $(i = 1, 2,\ k \in Z^+)$ are nonnegative constants. Suppose there exists $\tau > 0$ such that $0 \leqslant r_i(t) \leqslant \tau$. Further assume

$$\big|b_1(t)\big| + \big|a_2(t)\big| \leqslant a_1(t) + \frac{1}{1+t^2}, \qquad \big|b_2(t)\big| + \big|c_1(t)\big| \leqslant c_2(t) + \frac{1}{1+t^2},$$

and for $k \in Z^+$, there exist $d_k \geqslant 0$, $\sum_{k=1}^{\infty} d_k < \infty$ such that

$$\max\big\{(d_{k_1} + d_{k_2})^2, (e_{k_1} + e_{k_2})^2\big\} \leqslant (1 + d_k).$$

If we choose $V_1(t) = x_1^2/2$, $V_2(t) = x_2^2/2$, then the assumption (1) in Theorem 1 is obviously satisfied. Moreover, when $V_1(t) \geqslant V_2(t)$ (i.e., $|x_1(t)| \geqslant |x_2(t)|$), we have

$$\begin{aligned} D^+V_1(t) &= -a_1(t)x_1^2(t) + a_2(t)x_1(t)x_2(t) + b_1(t)x_1(t)x_1\big(t - r_1(t)\big) \\ &\leqslant -\big[a_1(t) - \big|a_2(t)\big| - \big|b_1(t)\big|\big]x_1^2(t) \leqslant \frac{2}{1+t^2}V_1(t), \end{aligned}$$

if $V_1(s) \leqslant V_1(t)$, $s \in [t - \tau, t]$ (i.e., $|x_1(t - r_1(t))| \leqslant |x_1(t)|$); when $V_1(t) \leqslant V_2(t)$ (i.e., $|x_1(t)| \leqslant |x_2(t)|$), we have

$$\begin{aligned} D^+V_2(t) &= c_1(t)x_1(t)x_2(t) - c_2(t)x_2^2(t) + b_2(t)x_2(t)x_2\big(t - r_2(t)\big) \\ &\leqslant -\big[c_2(t) - \big|c_1(t)\big| - \big|b_2(t)\big|\big]x_2^2(t) \leqslant \frac{2}{1+t^2}V_2(t), \end{aligned}$$

if $V_2(s) \leqslant V_2(t)$, $s \in [t - \tau, t]$ (i.e., $|x_2(t - r_2(t))| \leqslant |x_2(t)|$); therefore, the assumption (ii) in Theorem 1 is satisfied. For any $k \in Z^+$, when $V_1(t_k^-) \geqslant V_2(t_k^-)$, i.e., $x_1^2(t_k^-) \geqslant x_2^2(t_k^-)$, we have

$$\begin{aligned} V_1(t_k) = x_1^2(t_k)/2 &= \big(d_{k_1}x_1\big(t_k^-\big) + d_{k_2}x_2\big(t_k^-\big)\big)^2/2 \\ &\leqslant (d_{k_1} + d_{k_2})^2 x_1^2\big(t_k^-\big)/2 \leqslant (1 + d_k)V_1\big(t_k^-\big) \end{aligned}$$

and

$$\begin{aligned} V_2(t_k) = x_2^2(t_k)/2 &= \big(e_{k_1}x_1\big(t_k^-\big) + e_{k_2}x_2\big(t_k^-\big)\big)^2/2 \\ &\leqslant (e_{k_1} + e_{k_2})^2 x_1^2\big(t_k^-\big)/2 \leqslant (1 + d_k)V_1\big(t_k^-\big). \end{aligned}$$

When $V_1(t_k^-) \leqslant V_2(t_k^-)$, i.e., $x_1^2(t_k^-) \leqslant x_2^2(t_k^-)$, we also have

$$\max\big\{V_1(t_k), V_2(t_k)\big\} \leqslant (1 + d_k)V_2\big(t_k^-\big).$$

Hence, the assumption (iii) in Theorem 1 is also satisfied.

It is easy to see that the equations

$$\begin{cases} y' = \dfrac{2}{1+t^2} y, & t \neq t_k, \\ y(t_k) = (1+d_k) y\big(t_k^-\big), & k \in Z^+, \end{cases} \tag{15}$$

have solution $y(t) = \prod_{t_0 < t_k \leqslant t} (1+d_k) y_0 e^{2\arctan t - 2\arctan t_0}$. So the zero solution $y(t) \equiv 0$ of (15) is uniformly stable and by Theorem 1 we can conclude the uniform stability of (14).

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